Lie Series Technique, Ordinary Differential Equations and Dynamical Integration

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The Lie series technique is applied to solve a class of nonlinear differential equations. We show that it includes also the dynamical integration technique. Differential equations with chaotic behaviour are also considered with this technique. Finally first integrals and the Lie series technique is discussed.

Key words: Differential Equations; Lie Series Technique; First Integrals.

The Lie series technique is a powerful tool to solve systems of linear and nonlinear ordinary differential equations [1, 2]. Consider, for example, the autonomous second-order ordinary differential equation

$$\frac{d^2x}{dt^2} + g(x)\frac{dx}{dt} + f(x) = 0,$$

where g and f are analytic functions. This equation includes many limit cycles systems, for example the von der Pol equation. The equation can be written as a system of first-order differential equation:

$$\frac{dx_1}{dt} = x_2,$$
$$\frac{dx_2}{dt} = -g(x_1)x_2 - f(x_1).$$

To solve this system, using the Lie series technique, we consider the vector field

$$V = V_1(x_1,x_2)\frac{\partial}{\partial x_1} + V_2(x_1,x_2)\frac{\partial}{\partial x_2} = x_2\frac{\partial}{\partial x_1} + (-g(x_1)x_2 - f(x_1))\frac{\partial}{\partial x_2},$$

given by this system of differential equations. Then for sufficiently small t the solution of the initial value problem is given by the Lie series

$$x_1(t) = e^{tV}x_1|_{x_1=x_1(0),x_2=x_2(0)},$$
$$x_2(t) = e^{tV}x_2|_{x_1=x_1(0),x_2=x_2(0)}. \quad (1)$$

In the standard approximation, e^{tV} is expanded up to a certain order in t [2]. If we expand e^{tV} up to t, we obtain Euler's method. If we expand it up to t^2, we obtain the map

$$x_1(t) = x_1(0) + tVx_1|_{x\to x(0)} + \frac{t^2}{2!}V(V(x_1))|_{x\to x(0)},$$
$$x_2(t) = x_2(0) + tVx_2|_{x\to x(0)} + \frac{t^2}{2!}V(V(x_2))|_{x\to x(0)}.$$

Here we consider the approximation

$$e^{tV} \equiv e^{t(V_1+V_2)} \approx e^{tV_1}e^{tV_2}. \quad (2)$$

This is an approximation, since

$$[tV_1tV_2] \equiv t^2[V_1,V_2] \neq 0$$

i. e., we do not expand e^{tV} with respect to t. We also consider extensions, where we take into account that $[V_1,V_2] \neq 0$. However, for sufficiently small t it provides a good enough approximation. Obviously the operator e^{tV} is linear. We have

$$e^{tV_2}x_1 = e^{t(-g(x_1)x_2-f(x_2))\partial/\partial x_2}x_1 = x_1,$$

$$e^{tV_1}e^{tV_2}x_1 = e^{tV_1}x_1 = e^{tx_2\partial/\partial x_1}x_1 = x_1 + tx_2,$$

$$e^{tV_2}x_2 = e^{t(-g(x_1)x_2-f(x_1))\partial/\partial x_2}x_2 = x_2e^{-tg(x_1)} + f(x_1)\frac{e^{-tg(x_1)}-1}{g(x_1)},$$

$$e^{tV_1}e^{tV_2}x_2 = x_2e^{-tg(x_1+tx_2)} + f(x_1+tx_2)\frac{e^{-tg(x_1+tx_2)}-1}{g(x_1+tx_2)}.$$

Here we used that if $h_1(x)$ and $h_2(x)$ are analytic functions we have

$$e^{tV}h_1(x)h_2(x) = h_1(e^{tV}x)h_2(e^{tV}x).$$

We can also write

$$e^{tV_1}e^{tV_2}x_2 = x_2e^{-tg(x_1-tx_2)} - tf(x_1+tx_2) \cdot \left(1 - \frac{t}{2!}g(x_1) + \frac{t^2}{3!}g^2(x_1) - \cdots\right).$$

Thus we obtain the map

$$x_1(t) = x_1(0) + tx_2(0),$$

$$x_2(t) = x_2(0)e^{-tg(x_1(t))} + f(x_1(t))\frac{(e^{-tg(x_1(t))}-1)}{g(x_1(t))},$$

or

$$x_1(t) = x_1(0) + tx_2(0),$$

$$x_2(t) = x_2(0)e^{-tg(x_1(t))} - tf(x_1(t)) \cdot \left(1 - \frac{t}{2!}g(x_1(t)) + \frac{t^2}{3!}g^2(x_1(t)) - \cdots\right).$$

Brown and Chua [3] used dynamical integration to solve (1). They provided the map

$$x_1(t) = x_1(0) + tx_2(0),$$

$$x_2(t) = x_2(0)e^{-tg(x_1(t))} - tf(x_1(t)).$$

This is a special case of the case given above, if we expand

$$f(x_1)\frac{e^{-tg(x_1)}-1}{g(x_1)} \approx -tf(x_1)$$

up the first order in t.

Consider the special case $g(x) = \varepsilon(x^2-1)$, $f(x) = x$ (van der Pol equation) with $\varepsilon > 0$. Then we obtain the map

$$x_1(t) = x_1(0) + tx_2(0),$$

$$x_2(t) = x_2(0)e^{-t\varepsilon(x_1^2(t)-1)} + x_1(t)\frac{(e^{-t\varepsilon(x_1^2(t)-1)}-1)}{\varepsilon(x_1^2(t)-1)}.$$

The van der Pol equation has only one fixed point, namely $(x_1^*, x_2^*) = (0,0)$. The map given above preserves this fixed point. If $\varepsilon > 0$, then the fixed point of the system of diffential equations is unstable. For the map we find the same behaviour. For the case $g(x) = \varepsilon$ and $f(x) = x$ we have a linear equation (damped harmonic oscillator) and the system of differential equations can be solved exactly using the Lie series technique, where $t \in [0,\infty)$.

Next we extend the differential equation described above to

$$\frac{d^2x}{dt^2} + g(x)\frac{dx}{dt} + f(x) = h(t),$$

where $h(t) = k\cos(\omega t)$ is an external force. We can write this equation as an autonomous system of the first-order differential equations

$$\frac{dx_1}{dt} = x_2,$$

$$\frac{dx_2}{dt} = -g(x_1)x_2 - f(x_1) + k\cos(x_3),$$

$$\frac{dx_3}{dt} = \omega.$$

Thus we have the vector field

$$\begin{aligned} V &= V_1(x_1,x_2,x_3)\frac{\partial}{\partial x_1} + V_2(x_1,x_2,x_3)\frac{\partial}{\partial x_2} + V_3(x_1,x_2,x_3)\frac{\partial}{\partial x_3} \\ &= x_2\frac{\partial}{\partial x_1} + (-g(x_1)x_2 - f(x_1))\frac{\partial}{\partial x_2} + \omega\frac{\partial}{\partial x_3} \end{aligned}$$

given by this system of differential equations. We find the commutators

$$\left[V_1\frac{\partial}{\partial x_1}, \omega\frac{\partial}{\partial x_3}\right] = 0$$

and

$$\left[V_2\frac{\partial}{\partial x_2}, \omega\frac{\partial}{\partial x_3}\right] = \omega k\sin(x_3)\frac{\partial}{\partial x_2}.$$

We use the approximation

$$e^{tV} \approx e^{tV_3}e^{tV_1}e^{tV_2}.$$

Straightforward calculations yield

$$e^{tV_2}x_1 = e^{t(-g(x_1)x_2 - f(x_2) + k\cos(x_3))\partial/\partial x_2}x_1 = x_1,$$

$$e^{tV_1}e^{tV_2}x_1 = e^{tV_1}x_1 = e^{tx_2\partial/\partial x_1}x_1 = x_1 + tx_2,$$

$$e^{tV_3}e^{tV_1}e^{tV_2}x_1 = x_1 + tx_2,$$

$$e^{tV_2}x_2 = e^{t(-g(x_1)x_2 - f(x_1) + k\cos(x_3))\partial/\partial x_2}x_2$$
$$= x_2 e^{-tg(x_1)} + f(x_1)\frac{e^{-tg(x_1)}-1}{g(x_1)} + k\cos(x_3)\frac{e^{tg(x_1)-1}}{g(x_1)},$$
$$e^{tV_1}e^{tV_2}x_2 = x_2 e^{-tg(x_1+tx_2)} + f(x_1+tx_2)\frac{e^{-tg(x_1+tx_2)}-1}{g(x_1+tx_2)} + k\cos(x_3)\frac{e^{tg(x_1+tx_2)-1}}{g(x_1+tx_2)},$$
$$e^{tV_3}e^{tV_1}e^{tV_2}x_2 = x_2 e^{-tg(x_1+tx_2)} + f(x_1+tx_2)\frac{e^{-tg(x_1+tx_2)}-1}{g(x_1+tx_2)} + k\cos(x_3+\omega t)\frac{e^{tg(x_1+tx_2)-1}}{g(x_1+tx_2)}.$$

Thus we obtain the map

$$x_3(t) = x_3(0) + \omega t,$$
$$x_1(t) = x_1(0) + tx_2(0),$$
$$x_2(t) = x_2(0)e^{-tg(x_1(t))} + f(x_1(t))\frac{(e^{-tg(x_1(t))}-1)}{g(x_1(t))} + k\cos(x_3(t))\frac{(e^{tg(x_1(t))}-1)}{g(x_1(t))}.$$

For $f(x) = x$ and $g(x) = \varepsilon(x^2-1)$ we have

$$x_3(t) = x_3(0) + \omega t,$$
$$x_1(t) = x_1(0) + tx_2(0),$$
$$x_2(t) = x_2(0)e^{-t\varepsilon(x_1^2(t)-1)} + x_1(t)\frac{(e^{-t\varepsilon(x_1^2(t)-1)}-1)}{\varepsilon(x_1^2(t)-1)} + k\cos(x_3(t))\frac{e^{t\varepsilon(x_1^2(t)-1)}-1}{\varepsilon(x_1^2(t)-1)}.$$

For the values $k = 5.0$, $a = 5.0$ and $\omega = 2.466$ there is numerical evidence [2,5] that the system shows chaotic behaviour.

The approximation can be improved if we take higher order commutators into account [2,4,6], for example

$$\exp(t(V_1+V_2)) = \exp\left(\frac{1}{2}tV_1\right)\exp(tV_2) \cdot \exp\left(\frac{1}{2}tV_1\right) + O(t^3). \quad (3)$$

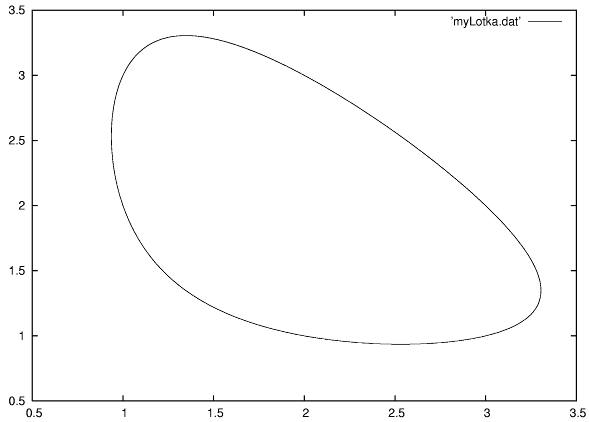

Fig. 1. Phase portrait in the $x_1 - x_2$ plane.

As an example, consider a Lotka-Volterra model with three species $(x_1, x_2, x_3 > 0)$:

$$\frac{dx_1}{dt} = x_1x_2 - x_1x_3,$$
$$\frac{dx_2}{dt} = x_2x_3 - x_1x_2,$$
$$\frac{dx_3}{dt} = x_3x_1 - x_2x_3.$$

It describes the interaction between three species, where species 1 feeds on species 2, species 2 feeds on species 3 and species 3 feeds on species 1. The model is of interest since it admits two first intgrals, namely $I_1(x) = x_1 + x_2 + x_3$ and $I_2(x) = x_1x_2x_3$. The fixed points of this system are the manifold $\{(x_1, x_2x_3) : x_1 = x_2 = x_3\}$. From the constants of motions $x_1 + x_2 + x_3 = C_1$ and $x_1x_2x_3 = C_2$, where $C_1 > 0, C_2 > 0$ and stability analysis we find that the system admits closed orbits as solutions. The map $x_j \to e^{tV_3}e^{tV_2}e^{tV_1}x_j (j = 1,2,3)$ preserves the fixed point of the original system of differential equations. The vector field corresponding to our Lotka-Volterra model is

$$V = (x_1x_2 - x_1x_3)\frac{\partial}{\partial x_1} + (x_2x_3 - x_1x_2)\frac{\partial}{\partial x_2} + (x_3x_1 - x_2x_3)\frac{\partial}{\partial x_3}.$$

Thus

$$e^{tV}(x_1 + x_2 + x_3) = x_1 + x_2 + x_3,$$
$$e^{tV}x_1x_2x_3 = x_1x_2x_3.$$

Let

$$V_1 = (x_1x_2 - x_1x_3)\frac{\partial}{\partial x_1},$$

$$V_2 = (x_2x_3 - x_1x_2)\frac{\partial}{\partial x_2},$$

$$V_3 = (x_3x_1 - x_2x_3)\frac{\partial}{\partial x_3}.$$

Then we find

$$e^{tV_1}e^{tV_2}e^{tV_3}(x_1+x_2+x_3) = (x_1+x_2+x_3)(1+O(t^2))$$

and

$$e^{tV_1}e^{tV_2}e^{tV_3}x_1x_2x_3 = (x_1x_2x_3)(1+O(t^2)),$$

since $[V_1,V_2] \neq 0$, $[V_1,V_3] \neq 0$ and $[V_2,V_3] \neq 0$. To obtain a better approximation, we have to take (3) into account. Figure 1 shows the phase portrait in the $x_1 - x_2$ plane, taking into account higher order commutators. The initial values are $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 3$.

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